CKM and PMNS Matrices

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Introduction

Most textbooks don't fully write how to arrive at the specific number of independent parameters in the CKM and PMNS matrices. There is some gesturing, but I've never seen an author show explicitly how to absorb arbitrary phases of the matrix into the definitions of the fields, so this note is to show how to do that.

Unitary Matrices

I think it makes sense that the CKM and PMNS matrices must be unitary. But how many independent parameters are in an arbitrary unitary matrix, and how many of these parameters are phases and how many are real amplitudes?

First, consider a matrix U in the defining representation of $\mathrm{GL}(n, \mathbb{C})$. This matrix has n^2 independent complex parameters, or $2n^2$ independent real parameters. When we impose the unitary constraint,

$$U^\dagger U = 1, \tag{1}$$

we have a matrix equation that reduces to n^2 complex equations, or $2n^2$ equations for the real parameters, n^2 for the real parts and n^2 for the imaginary parts. From experience, these $2n^2$ equations must not all be independent, since they would otherwise fully constrain U. How many are independent then?

We can consider each element of $U^\dagger U$ as an equation. The fact that the identity matrix is Hermitian tells us

$$\left(U^\dagger U\right)^\dagger = 1^\dagger = 1 \implies \left(U^\dagger U\right)^\dagger = U^\dagger U. \tag{2}$$

The diagonal elements of $U^\dagger U$ must therefore be real. Also, each element in the upper-right triangle is related to an element in the lower-right triangle by

$$\left(U^\dagger U\right)_{ab} = \left(U^\dagger U\right)^*_{ba}. \tag{3}$$

Therefore, the independent parts of $U^\dagger U$ can be chosen to be the diagonal and upper-right triangle. The diagonal is described by n real parameters since these elements are real, and the upper-right triangle is in general complex, so it's described by $(n^2 - n)/2$ complex parameters or $n^2 - n$ real parameters. The total number of real parameters to describe $U^\dagger U$ is then $n^2 - n + n = n^2$.

Since there are n^2 independent parameters in $U^\dagger U$, this tells us we have n^2 constraints on the $2n^2$ real parameters of U. Therefore, U has $2n^2 - n^2 = n^2$ real independent parameters. The elements of U are in general complex, they can be described by $Ae^{i\phi}$, so our real parameters can either be the amplitudes A or phases ϕ.

How many of these n^2 parameters are amplitudes or phases? The easiest way to see this is to repeat the analysis for orthogonal matrices. Suppose that all the $\phi = 0$. Then $U^*_{ab} = U_{ab}$. The unitary condition then reduces to

$$\delta_{ac} = (U^\dagger)_{ab} U_{bc} = (U^T)^*_{ab} U_{bc} \to \delta_{ac} = (U^T)_{ab} U_{bc}, \tag{4}$$

where $U^T U = 1$ defines an orthogonal matrix U. So, if we find out how many real parameters describe an n-by-n orthogonal matrix, we find how many of the n^2 parameters of our unitary matrix are amplitudes.

Let O be a real matrix subject to $O^T O = 1$. At most there can be n^2 free parameters. Since the identity is symmetric, $1 = 1^T$, we have a similar relation:

$$(O^T O)^T = O^T O. \tag{5}$$

Each element of $O^T O$ represents a constraint equation. All n entries on the diagonal are independent, since they must only equal themselves which is trivially true. The upper-right triangle is related to the lower-left triangle by $(O^T O)_{ab} = (O^T O)_{ba}$, so we can choose the upper-right triangle as having the independent elements. Therefore, the number of free parameters in $O^T O$ is $(n^2 - n)/2 + n = (n^2 + n)/2$. This is the number of constraints on O, so the total number of free parameters in O is $n^2 - (n^2 + n)/2 = (n^2 - n)/2$.

So, returning to the unitary n-by-n matrix, of the n^2 real parameters, $(n^2 - n)/2$ must be amplitudes which leaves a leftover $(n^2 + n)/2$ as phases.

The CKM Matrix

The Cabibbo-Kobayashi-Maskawa matrix encodes the coupling between quark generations under the weak interaction. We won't prove it here, but one can check (e.g. in Schwartz) that in the mass basis, the weak interaction has terms like

$$W^+_\mu \bar{u}^i_L \gamma^\mu V^{ij} d^j_L \tag{6}$$

with V the CKM matrix and u_L and d_L the left-handed up-type and down-type flavor multiplets, respectively (we use the term multiplet to leave as a variable the number of flavor generations).

The gamma matrix is not important because it only mixes the components of a single spinor in the up- or down-type flavor multiplets. So, we can restrict our view to just investigating

$$\bar{u}_L V d_L. \tag{7}$$

In addition, we assume each up- and down-type multiplet retains a $U(1)$ symmetry for each component.

The textbook statement is that we are allowed to alter $2n$ phases (since there are $2n$ fields); however, if the phases are chosen to all be the same magnitude, it doesn't affect V. Therefore, we may only eliminate $2n - 1$ phases from V via redefinitions of the fields. The CKM matrix for n generations thus has $(n^2 + n)/2 - (2n + 1) = (n - 2)(n - 1)/2$ phases for a total of $(n - 1)^2$ total free parameters. Let's investigate this more closely.

Two Generations

This was the original formulation of this matrix, and where we get the Cabibbo angle. A general 2-by-2 unitary matrix can be put in the form:

$$V = \begin{bmatrix} Ae^{ia} & Be^{ib} \\ -Be^{-ib}e^{ic} & Ae^{-ia}e^{ic} \end{bmatrix} \tag{8}$$

with the condition $A^2 + B^2 = 1$. One can check that such a matrix is indeed unitary. We next want to redefine the fields such that we can eliminate the different phases. Multiplying everything out we get

$$\begin{aligned} \begin{bmatrix} \bar{u} & \bar{c} \end{bmatrix} \begin{bmatrix} Ae^{ia} & Be^{ib} \\ -Be^{-ib}e^{ic} & Ae^{-ia}e^{ic} \end{bmatrix} \begin{bmatrix} d \\ s \end{bmatrix} &= \begin{bmatrix} \bar{u} & \bar{c} \end{bmatrix} \begin{bmatrix} Ae^{ia}d + Be^{ib}s \\ -Be^{-ib}e^{ic}d + Ae^{-ia}e^{ic}s \end{bmatrix} \\ &= Ae^{ia}\bar{u}d + Be^{ib}\bar{u}s - Be^{-ib}e^{ic}\bar{c}d + Ae^{-ia}e^{ic}\bar{c}s \\ &= A(e^{ia/2}e^{ib/2}\bar{u})(e^{ia/2}e^{-ib/2}d) + B(e^{ia/2}e^{ib/2}\bar{u})(e^{-ia/2}e^{ib/2}s) \\ &\quad - B(e^{-ia/2}e^{-ib/2}e^{ic}\bar{c})(e^{ia/2}e^{-ib/2}d) + A(e^{-ia/2}e^{-ib/2}e^{ic}\bar{c})(e^{-ia/2}e^{ib/2}s) \end{aligned} \tag{9}$$

So, if we make the simultaneous redefinitions

$$\begin{aligned} \bar{u} &\to e^{-ia/2}e^{-ib/2}\bar{u} \\ \bar{c} &\to e^{ia/2}e^{ib/2}e^{-ic}\bar{c} \\ d &\to e^{-ia/2}e^{ib/2}d \\ s &\to e^{ia/2}e^{-ib/2}s, \end{aligned} \tag{10}$$

we see that we can cancel all three phases, and the previous equation becomes

$$\begin{bmatrix} \bar{u} & \bar{c} \end{bmatrix} \begin{bmatrix} Ae^{ia} & Be^{ib} \\ -Be^{-ib}e^{ic} & Ae^{-ia}e^{ic} \end{bmatrix} \begin{bmatrix} d \\ s \end{bmatrix} = A\bar{u}d + B\bar{u}s - B\bar{c}d + A\bar{c}s = \begin{bmatrix} \bar{u} & \bar{c} \end{bmatrix} \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \begin{bmatrix} d \\ s \end{bmatrix} \tag{11}$$

Since $A^2 + B^2 = 1$, we can parameterize A and B by $A = \cos\theta$, $B = \sin\theta$, where θ is the Cabibbo angle. Therefore, the CKM matrix for two flavor generations is

$$V = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \tag{12}$$

with $(2-1)^2 = 1$ free parameter, θ. The PDG (rev. 2021) gives the most precise determination of $\sin\theta = 0.2277(13)$, with data from superallowed nuclear beta decays.

Three Generations

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CP Violation

I now want to prove that for ≥ 3 generations of quarks, the weak interaction in general will violate CP. Schwartz gives a nice definition of the interaction Lagrangian, but I want to use Weinberg's gamma matrices, so I must convert into Weinberg's metric diag$(-,+,+,+)$. After conversion, the interaction Lagrangian looks like:

$$\mathscr{L}_{\text{int}} = -\frac{e}{\sqrt{2}\sin(\theta_W)} \left(W_\mu^+ \bar{u}_L^i \gamma^\mu V^{ij} d_L^j + W_\mu^- \bar{d}_L^j \gamma^\mu (V^\dagger)^{ji} u_L^i \right) \tag{13}$$

where $i = 1, 2, ..., N$ for N generations of quarks. The V just mix the multiplet components, so we focus on how

$$W^+_\mu \bar{u}^i_L \gamma^\mu d^j_L = W^+_\mu \bar{u}^i \gamma^\mu P_L d^j \tag{14}$$

transforms under CP where P_L is the left-chiral projector

$$P_L = \frac{1+\gamma_5}{2} \tag{15}$$

with Weinberg's definition of the various gamma matrices

$$\gamma^0 = -i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \vec{\gamma} = -i \begin{bmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{bmatrix}, \gamma_5 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}. \tag{16}$$

We need to recall Weinberg's definitions of how C and P act on spinor fields. He gives:

$$\begin{aligned} \mathsf{P}\psi_l \mathsf{P}^{-1} &= \eta^* \beta_{lm} \psi_m \\ \mathsf{C}\psi_l \mathsf{C}^{-1} &= -\xi^* \beta_{lm} \mathscr{C}_{mn} \psi^*_n \end{aligned} \tag{17}$$

where we've dropped the position dependence since it doesn't matter, the η, ξ are the intrinsic P- and C-parities of the particles the fields annihilate, and ψ^* means take the Hermitian conjugate of the component, and don't worry about transposing between row and column vectors. We also have the definition of the $\mathscr{C}$ matrix

$$\mathscr{C} = \gamma^2 \beta = -i \begin{bmatrix} \sigma_2 & 0 \\ 0 & -\sigma_2 \end{bmatrix}. \tag{18}$$

which one can check satisfy the following identities:

$$\mathscr{C}^\dagger = \mathscr{C}^T = \mathscr{C}^{-1} = -\mathscr{C}, \quad \mathscr{C}^* = \mathscr{C}. \tag{19}$$

We see that the action of CP is thus

$$\begin{aligned} \mathsf{CP}\psi_l(\mathsf{CP})^{-1} &= \eta^* \beta_{lm} \mathsf{C}\psi_m \mathsf{C}^{-1} \\ &= -\eta^* \xi^* \beta_{lm} \beta_{mn} \mathscr{C}_{na} \psi^*_a \\ &= \eta^* \xi^* \psi^*_m \mathscr{C}_{ml} \\ &= \eta^* \xi^* \bar{\psi}_n \beta_{nm} \mathscr{C}_{ml} \end{aligned} \tag{20}$$

where we used the identities $\mathscr{C}^T = -\mathscr{C}$ and $\beta^T = \beta$.

The action of CP on the Dirac adjoint is

$$\begin{aligned} \mathsf{CP}\bar{\psi}_l(\mathsf{CP})^{-1} &= \mathsf{CP}\psi^*_m(\mathsf{CP})^{-1}\beta_{ml} \\ &= \eta\xi\psi_n \mathscr{C}^*_{nm} \beta_{ml} \\ &= \eta\xi\psi_n \mathscr{C}_{nm} \beta_{ml} \\ &= -\eta\xi \beta_{lm} \mathscr{C}_{mn} \psi_n \\ &= +\eta\xi \mathscr{C}_{lm} \beta_{mn} \psi_n \\ &= -\eta\xi (\mathscr{C}^{-1})_{lm} \beta_{mn} \psi_n \end{aligned} \tag{21}$$

where we used the identities $\mathscr{C}^* = \mathscr{C}$, $\mathscr{C}^T = -\mathscr{C}$, $\beta^T = \beta$, and $\beta\mathscr{C} = -\mathscr{C}\beta$.

We should also note

$$P_L^T = P_L, \quad \beta\mathscr{C}P_L = P_R \beta\mathscr{C}, \tag{22}$$

where the last identity came from the fact we commuted three gamma matrices past the P_L, thus changing sign on the γ_5 and swapping P_L to P_R.

The final thing we need to do is rewrite one of Weinberg's identities:

$$\gamma_\mu^* = \beta\mathscr{C}\gamma_\mu\mathscr{C}^{-1}\beta \implies \gamma_\mu^\dagger = \beta^T(\mathscr{C}^{-1})^T\gamma_\mu^T\mathscr{C}^T\beta^T = \beta\mathscr{C}\gamma_\mu^T\mathscr{C}^{-1}\beta \tag{23}$$

By inspection of the form of the gamma matrices, the Hermitian conjugate can be related to the space inversion transformation $\mathscr{P}^\mu{}_\nu = \mathrm{diag}(1,-1,-1,-1)$:

$$(\gamma^\mu)^\dagger = -\mathscr{P}^\mu{}_\nu\gamma^\nu \tag{24}$$

Finally, we can compute how CP acts on the charged current

$$\begin{aligned}\mathsf{CP}\bar{u}^i\gamma^\mu P_L d^j(\mathsf{CP})^{-1} &= \mathsf{CP}\bar{u}^i_l(\mathsf{CP})^{-1}\gamma^\mu_{lm}(P_L)_{mn}\mathsf{CP}d^j_n(\mathsf{CP})^{-1}\\ &= -|\eta|^2|\xi|^2(\mathscr{C})^{-1}_{lc}\beta_{cd}u^i_d\gamma^\mu_{lm}(P_L)_{mn}\bar{d}^j_a\beta_{ab}\mathscr{C}_{bn}\\ &= +\bar{d}^j_a\beta_{ab}\mathscr{C}_{bn}\gamma^\mu_{lm}(P_L)_{mn}(\mathscr{C})^{-1}_{lc}\beta_{cd}u^i_d\\ &= +\bar{d}^j_a\beta_{ab}\mathscr{C}_{bn}(P_L)_{nm}(\gamma^{\mu T})_{ml}(\mathscr{C})^{-1}_{lc}\beta_{cd}u^i_d\\ &= +\bar{d}^j_a(P_R)_{ab}\beta_{bn}\mathscr{C}_{nm}(\gamma^{\mu T})_{ml}(\mathscr{C})^{-1}_{lc}\beta_{cd}u^i_d\\ &= -\mathscr{P}^\mu{}_\nu\bar{d}^j_a(P_R)_{ab}\gamma^\mu_{bc}u^i_c\\ &= -\mathscr{P}^\mu{}_\nu\bar{d}^j\gamma^\mu P_L u^i\end{aligned} \tag{25}$$

where in going from the second line to the third line, we pick up a $-$ sign because we are moving two fermionic operators past each other. We don't pick up any extra terms, because by definition, we assume this interaction is normal-ordered.

Next, how does the W boson transform under CP? From Weinberg, we see

$$\mathsf{CP}W^+_\mu(\mathsf{CP})^{-1} = -\eta^*\xi^*\mathscr{P}_\mu{}^\nu W^-_\nu \tag{26}$$

where η, and ξ are the intrinsic P- and C-parities of the W^+, now. Since the W bosons are not eigenstates of either C or P, by convention we have $\eta=-1$, $\xi=-1$. Therefore, the contracted quantity transforms like

$$\mathsf{CP}W^+_\mu\bar{u}^i\gamma^\mu P_L d^j(\mathsf{CP})^{-1} = \mathscr{P}_\mu{}^\nu W^-_\nu\mathscr{P}^\mu{}_\rho\bar{d}^j\gamma^\rho P_L u^i = W^-_\mu\bar{d}^j\gamma^\mu P_L u^i. \tag{27}$$

One can also show that the second term in the interaction transforms in a similar way, namely

$$\mathsf{CP}W^-_\mu\bar{d}^j_L\gamma^\mu u^i(\mathsf{CP})^{-1} = W^+_\mu\bar{u}^i_L\gamma^\mu d^j_L. \tag{28}$$

The full interaction term then transforms like

$$\begin{aligned}\mathsf{CP}\mathscr{L}_{\text{int}}(\mathsf{CP})^{-1} &= \mathsf{CP}\left(-\frac{e}{\sqrt{2}\sin(\theta_W)}\left(W^+_\mu\bar{u}^i_L\gamma^\mu V^{ij}d^j_L + W^-_\mu\bar{d}^j_L\gamma^\mu(V^\dagger)^{ji}u^i_L\right)\right)(\mathsf{CP})^{-1}\\ &= -\frac{e}{\sqrt{2}\sin(\theta_W)}\left(W^+_\mu\bar{u}^i_L\gamma^\mu(V^\dagger)^{ji}d^j_L + W^-_\mu\bar{d}^j_L\gamma^\mu V^{ij}u^i_L\right)\end{aligned} \tag{29}$$

So, the condition that the interaction is invariant under CP is that

$$(V^\dagger)^{ji} = V^{ij} \implies V^* = V, \tag{30}$$

that is, the CKM matrix is real.

As we've shown in the previous sections, for ≥ 3 generations of quarks, there will in general be at least one non-zero phase, such that $V^* \neq V$, and therefore, CP violation is possible.